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Probability densities of periodically driven noisy systems: An approximation scheme incorporating linear-response and adiabatic theory

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We consider periodically driven noisy systems in the limit of long times. To deduce their asymptotic time-periodic probability distributions, two approaches are commonly used: adiabatic theory, valid if driving is very slow, and linear-response theory, applicable when driving is weak. We introduce an approximation scheme that combines these two approaches to yield the driven probability distribution even when driving is strong and moderately fast, so that both linear-response and adiabatic approximations break down. The high accuracy of this scheme is demonstrated on a driven overdamped noisy oscillator in a bistable quartic potential.

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In many natural and experimental situations, the system of interest experiences a combined action of a time-periodic deterministic driving and noise. The former may be imposed by an experimentalist or some other external agent, while the latter results from a huge number of uncontrolled microscopic degrees of freedom coupled to the system. Specific examples can be found in areas as diverse as ion transport in molecular pumps [1], atomic friction [2,3], dielectric response [4,5], climatology [6,7], etc. Furthermore, in contrast to the deterministic case, noise can change the response and transport properties of a system, sometimes introducing surprising new effects, such as stochastic resonance [8], the ratchet effect [9], synchronization [10], and absolute negative mobility [11]. For these reasons, the subject of periodically driven noisy systems constitutes an important part of nonequilibrium statistical physics dealt with in an increasing number of articles [3,5–7,10–21,21–27] and review papers [8,9,28,29].

The evolution of the probability density $W(x,t)$ to find the system in the state x (in general, a vector) at time t is governed by the Fokker-Planck equation (FPE) [30]

$$\partial W(x,t)/\partial t = \hat{L}_0(x)W(x,t) + a(t)\hat{L}_1(x)W(x,t), \qquad (1)$$

where the operator $\hat{L}_0$ describes the undriven system and $\hat{L}_1$ the effect of the time-periodic driving field $a(t)$ with amplitude a_0 and frequency Ω. In the following, we will focus on the systems, whose probability distribution in the long-time limit is a periodic function of time, oscillating with the frequency of external forcing,

$$W(x,t+2\pi/\Omega) = W(x,t). \qquad (2)$$

Determination of this asymptotic distribution is the principal task of this work.

In general, an exact treatment of this problem is possible only in a rather limited number of special cases [13–16]. Therefore, various analytic approximations have been introduced, which make use of the smallness of some parameter, such as driving amplitude [5,17–20,29], frequency [12,21–23], period [24,25], or noise level [3,26,27]. In contrast, our goal is to develop an approximation that is valid in the challenging regime where *no small parameter is present*.

Although the approximation scheme described below is rather general and applies to a variety of systems, we will focus here on one paradigmatic example, namely a driven overdamped noisy oscillator, which finds itself in a confining nonlinear time-dependent potential $U(x)-a(t)x$. The dynamics of the system's coordinate x is governed by the Langevin equation $\partial x/\partial t = -U'(x)+a(t)+\sqrt{2T}\xi(t)$, where $\xi(t)$ is unbiased Gaussian white noise of unit strength and T is the temperature. The dynamics of the coordinate probability distribution $W(x,t)$ is described by the Fokker-Planck equation (1) with the operators given by [30]

$$\hat{L}_0(x)W(x,t) = \frac{\partial}{\partial x}\left(T\frac{\partial W}{\partial x} + U'(x)W\right),$$

$$\hat{L}_1(x)W(x,t) = -\frac{\partial W}{\partial x}. \qquad (3)$$

Within the adiabatic approximation, the driven probability distribution at each moment of time is completely determined by the instantaneous force value $a(t)$,

$$W_A(x,t) = \frac{\exp\{-[U(x)-a(t)x]/T\}}{\int_{-\infty}^{\infty} dx' \exp\{-[U(x')-a(t)x']/T\}}. \qquad (4)$$

This expression is valid only if driving is adiabatically slow, so that the system is always in the accompanying equilibrium with the instantaneous driving. At the same time, the driving amplitude a_0 can be arbitrarily large.

Linear-response theory, on the other hand, applies to the case of very small driving amplitude, a_0, without any restrictions on driving frequency. The linear-response approximation for the driven probability distribution is written as the first-order polynomial in a_0,

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$$W_L(x,t) = W_{\mathrm{eq}}(x)[1 + a_0 w(x,t)],$$

$$W_{\mathrm{eq}}(x) = \frac{\exp[-U(x)/T]}{\int_{-\infty}^{\infty} dx' \exp[-U(x')/T]}. \quad (5)$$

Furthermore, normalization of $W_L(x,t)$ to 1 implies that the linear-response correction $w(x,t)$ has the property

$$\langle w(x,t)\rangle_{\mathrm{eq}} \equiv \int_{-\infty}^{\infty} dx W_{\mathrm{eq}}(x) w(x,t) = 0. \quad (6)$$

Although the adiabatic limit (4) of the driven probability distribution depends exponentially strongly on the amplitude of driving, the argument of the exponentials in the numerator and denominator is only a linear function of a_0. This means that if one writes the driven probability density in the form $W(x,t) = e^{f(x,t)}/\int dx' e^{f(x',t)}$ and approximates the unknown function $f(x,t)$ linearly in a_0, so as to reproduce the linear-response result (5) correctly, then this approximation becomes exact in the limit of adiabatically slow driving. Furthermore, this approximation may be expected to work reasonably well even at finite driving frequencies, because it takes into account the effect of retardation between the driving and the system, as the linear-response formula (5) does. With this in mind, we arrive at the following *lineabatic* approximation (LA) for the driven probability density:

$$W_{\mathrm{LA}}(x,t) = \frac{\exp[-U(x)/T + a_0 w(x,t)]}{\int_{-\infty}^{\infty} dx' \exp[-U(x')/T + a_0 w(x',t)]}. \quad (7)$$

This expression is the main conceptual step of the present work.

The remaining practical task is to determine the linear-response correction $w(x,t)$ for the system described by Eqs. (1) and (3). For definiteness, we assume the driving to be harmonic,

$$a(t) = a_0 \cos(\Omega t). \quad (8)$$

Upon substitution of the linear-response ansatz (5) into the FPE (1) and neglecting the terms $O(a_0^2)$, we get

$$\partial w(x,t)/\partial t = \hat{L}_0^{\dagger} w(x,t) + \cos(\Omega t)[W_{\mathrm{eq}}(x)]^{-1} \hat{L}_1 W_{\mathrm{eq}}(x), \quad (9)$$

where the adjoint (or backward) Fokker-Planck operator is defined by [cf. Eq. (3)]

$$\hat{L}_0^{\dagger} f(x) = T f''(x) - U'(x) f'(x). \quad (10)$$

Next, we Fourier-decompose $w(x,t)$ as

$$w(x,t) = \frac{1}{2}[\tilde{w}(x,\Omega) e^{i\Omega t} + \tilde{w}^*(x,\Omega) e^{-i\Omega t}] \quad (11)$$

with the complex-valued function $\tilde{w}(x,\Omega)$ satisfying the equation

$$i\Omega \tilde{w}(x,\Omega) = \hat{L}_0^{\dagger} \tilde{w}(x,\Omega) + [W_{\mathrm{eq}}(x)]^{-1} \hat{L}_1 W_{\mathrm{eq}}(x). \quad (12)$$

The solution of Eq. (12) is sought as a power series of $i\Omega$,

$$\tilde{w}(x,\Omega) = \sum_{n=0}^{\infty} (i\Omega)^n g_n(x) \quad (13)$$

with the expansion coefficients $g_n(x)$ satisfying the recursive hierarchy of equations for $n = 0, 1, 2, \ldots$,

$$\hat{L}_0^{\dagger} g_n = g_{n-1}, \quad g_{-1}(x) := -(W_{\mathrm{eq}})^{-1} \hat{L}_1 W_{\mathrm{eq}}(x). \quad (14)$$

For the operator (10), each term of this hierarchy can be found in quadratures,

$$g_n(x) = \frac{1}{T}\int_0^x dy \int_{-\infty}^{y} dz e^{[U(y)-U(z)]/T} g_{n-1}(z) + g_n(0), \quad (15)$$

where the integration constant $g_n(0)$ is determined from the condition

$$\langle g_n(x)\rangle_{\mathrm{eq}} = 0, \quad (16)$$

which follows from Eq. (6).

For practical evaluation of the function $\tilde{w}(x,\Omega)$, one needs to truncate the power series (13) at some value of the summation index n. Direct use of the expansion (13) requires a very large number of terms for convergence. To accelerate the convergence, we note the following.

Equation (12) has a formal solution $\tilde{w}(x,\Omega) = (i\Omega - \hat{L}_0^{\dagger})^{-1} W_{\mathrm{eq}}^{-1} \hat{L}_1 W_{\mathrm{eq}}(x)$, which decays as $1/\Omega$ at high frequencies. This observation suggests that instead of the power series (13), it is more advantageous to use an $[N/(N+1)]$ Padé approximation, i.e., a rational function of $i\Omega$,

$$\tilde{w}_{\mathrm{Padé}}(x,\Omega) = \frac{\sum_{n=0}^{N} p_n(x)(i\Omega)^n}{1 + \sum_{m=1}^{N+1} p_{N+m}(x)(i\Omega)^m}, \quad (17)$$

where the $2N+1$ coefficients $p_n(x)$ are found from the requirement that the Taylor expansion in $i\Omega$ of the approximation (17) had the same $2N+1$ first terms as the power series (13). Our detailed investigation has indicated that even with $N=0$, Eq. (17) is a fair approximation to $\tilde{w}(x,\Omega)$, while the convergence of the Padé approximation (17) is already achieved at a relatively small value of $N=2$. We note that a similar method based on the Padé expansion has been introduced some time ago [31,32] to study relaxation processes in the absence of driving. Having determined the function $\tilde{w}(x,\Omega)$, the linear-response correction to the driven probability density (11) and the lineabatic driven distribution (7) follow readily.

A criterion of validity of the lineabatic ansatz can be established by considering the second-order correction to the driven probability density, presented as

$$W(x,t) = W_{\mathrm{eq}}(x)[1 + a_0 w(x,t) + a_0^2 u(x,t)] + O(a_0^3). \quad (18)$$

In the spirit of the previous discussion, we modify the adiabatic formula (4) so as to correctly reproduce the first two orders of the expansion (18), leading, up to a normalization constant, to the second-order lineabatic approximation

$$W_{\mathrm{LA2}}(x,t) \propto \exp(-U(x)/T + a_0 w(x,t) + a_0^2\{u(x,t) - [w^2(x,t) - \langle w^2(x,t)\rangle_{\mathrm{eq}}]/2\}). \quad (19)$$

Its comparison with the lineabatic ansatz (7) shows that the latter is a good approximation, provided that

$$|w(x,t)| \gg a_0 |u(x,t) - [w^2(x,t) - \langle w^2(x,t)\rangle_{\mathrm{eq}}]/2|. \quad (20)$$

This is to be compared with the condition of validity of the linear-response formula (5), namely

$$|w(x,t)| \gg a_0 |u(x,t)|, \quad (21)$$

which follows from Eqs. (5) and (18).

It can be shown that at sufficiently high frequencies, the right-hand sides of both inequalities (20) and (21) decay as $1/\Omega^2$, while the left-hand side drops only as $1/\Omega$. Therefore, both the linear-response formula (5) and the lineabatic approximation (7) are expected to work well at high frequencies, even for high driving amplitudes.

At low Ω, on the other hand, the situation is drastically different. In this case, the higher-order terms neglected in the linear approximation (5) are actually not at all small even for a small but finite amplitude a_0, because the function $w(x,t)$ can take on such values that the expression in the brackets of Eq. (5) becomes negative; then, the higher-order corrections must be of similar magnitude in order for the probability density $W(x,t)$ to remain positive at all values of x. Therefore, the criterion (21) of validity of linear response theory breaks down at low frequencies. In contrast, the expression on the right-hand side of the inequality (20) goes to zero as $\Omega \to 0$, and the lineabatic formula (7) reduces to the adiabatic one (4). Moreover, for a harmonic potential $U(x)$, the right-hand side of the criterion (20) is identically zero at all frequencies, and the lineabatic approximation (7) is exact and coincides with the result of Ref. [13]. For other potentials, the criterion (20) is satisfied for those values of Ω that are smaller than the smallest characteristic frequency of the system. Since the condition (20) of validity of the lineabatic approximation (7) holds at low and high Ω, one can expect that this approximation scheme will also work well at intermediate driving frequencies.

To demonstrate its accuracy in comparison with linear-response theory (5) and adiabatic theory (4), we apply these three approaches to the oscillator (1), (3), and (8) in a bistable potential,

$$U(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \quad (22)$$

In addition to these three analytic approximations, we have solved the Fokker-Planck equation (1) numerically. Our main interest is the behavior of the average coordinate $\langle x(t)\rangle \equiv \int dx x W(x,t)$, a quantity of interest in many contexts from atomic friction to stochastic resonance [2–8]; it is plotted versus instantaneous forcing (8) as hysteresis loops in Fig. 1.

These results are obtained for a relatively strong driving of the amplitude $a_0=0.5$; for comparison, the potential $U(x)-ax$ becomes monostable at external bias $a=2/\sqrt{27} \approx 0.385$. The slowest time scale characterizing the system in the potential (22) is the rate of thermally activated interwell hopping in the absence of driving. This quantity is given by Kramers' formula [33],

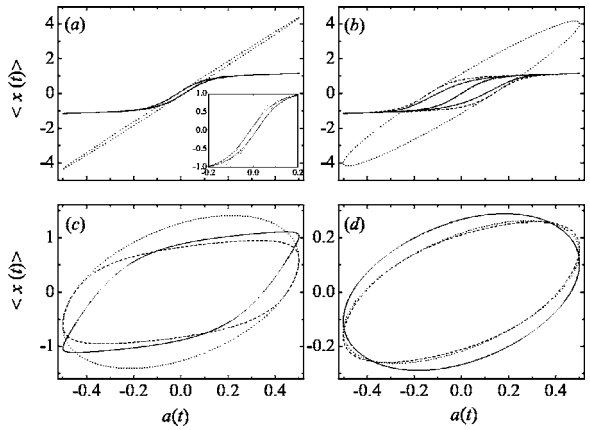

FIG. 1. Hysteresis loops depicting the relation between the average coordinate $\langle x(t)\rangle$ of the driven system (1) and (3) in the potential (22) and the driving (8) in the long-time limit. The temperature is $T=0.1$, driving amplitude $a_0=0.5$, and driving frequencies Ω are 10^{-3} (a), 10^{-2} (b), 0.1 (c), and 1 (d). Solid lines: numerical solution of the Fokker-Planck equation (1). Dotted lines: linear-response approximation (5). Dashed lines: lineabatic approximation (7) and (17) with $N=2$. Since the adiabatic approximation (4) is the same for all frequencies, it is depicted as a dashed-dotted line only in (b).

$$\omega_{\mathrm{Kr}} = \frac{\sqrt{|U''_{\min} U''_{\max}|}}{2\pi} \exp(-\Delta U/T), \quad (23)$$

where $U''_{\min,\max}$ refer to the second derivatives of the potential (22) at its extrema $x_{\min}=\pm 1$, $x_{\max}=0$, and $\Delta U=1/4$ is the height of the barrier separating the two minima. The calculations were performed for the temperature value $T=0.1$, at which Kramers' rate is $\omega_{\mathrm{Kr}} \approx 0.02$.

It is seen from Fig. 1(a) that at a small frequency $\Omega=0.001 \ll \omega_{\mathrm{Kr}}$, the linear-response theory yields the correct width of the hysteresis loop, but severely overestimates the amplitude of the driven oscillations of the average coordinate, while the hysteresis trace obtained from the lineabatic approximation is practically indistinguishable from the one obtained numerically [see the inset in Fig. 1(a)].

At driving frequency of the order of Kramers' rate, $\Omega=0.01 \approx \omega_{\mathrm{Kr}}$, the system cannot follow the relatively fast temporal variations of external forcing as effectively, leading to a retardation between the driven value of the average coordinate and external forcing and, consequently, to a broadening of the hysteresis loop. This effect is correctly reproduced by both the linear-response theory and lineabatic approximation, Fig. 1(b), although, again, the former overestimates the amplitude of driven vibrations by a factor of 4. The true width of the hysteresis loop is somewhat smaller than that predicted by both linear-response theory and our lineabatic approximation. The reason for this is as follows. The time scale necessary for the system to come into equilibrium with the instantaneous external forcing $a(t)$ changes periodically in time together with this forcing: the larger the value of external bias a, the faster the system comes to equi-

librium with it. Therefore, the retardation between driving and the system's average coordinate at large values of $a(t)$ is actually smaller than that predicted by the linear-response theory (5), and consequently the true width of the hysteresis loop is also smaller.

At still larger driving frequencies, the hysteresis loop becomes more rounded, see Fig. 1(c), and degenerates into an ellipse, Fig. 1(d), at Ω of the order of the intrawell frequencies.

In conclusion, we would like to note that the systematic improvement of the accuracy of lineabatic approximation (7) evidently should proceed by incorporating the higher-order terms of the expansion (18) in a way similar to Eq. (19). However, our additional analysis based on numerical evaluation of the function $u(x,t)$ has revealed that Eq. (19) is *not* more accurate than the lineabatic expression (7) outside of its range of validity specified by the inequality (20). The nontrivial task of finding better alternatives to Eq. (19) is the subject of our current research.

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